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Quasilinearization and Boundary Value Problems at Resonance for Caputo Fractional Differential Equations

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Abstract

The quasilinearization method is applied to a boundary value problem at resonance for a Caputo fractional differential equation. The method of upper and lower solutions is first employed to obtain the uniqueness of solutions of the boundary value problem at resonance. The shift argument is applied to show the existence of solutions. The quasilinearization algorithm is then developed and sequences of approximate solutions are constructed that converge monotonically and quadratically to the unique solution of the boundary value problem at resonance. Two applications are provided to illustrate the main results.
AMS Subject Classification: 26A33, 34K10, 34A45, 47H05

Key words: boundary value problem at resonance, Caputo fractional differential equations, shift method, upper and lower solutions, quasilinearization.

1 Introduction

The method of quasilinearization, introduced by Bellman [5, 6] in the 1960s, offers a numerical method to approximate solutions of nonlinear problems with sequences of solutions of linear problems. Under suitable hypotheses, the sequences of approximate solutions converge monotonically and quadratically. In the case of boundary value problems for ordinary differential equations, under modest hypotheses, the approximate hypotheses converge to a unique solution.

Quasilinearization proved to be useful in the study of initial value problems for ordinary differential equations (see, [17, 18, 19, 23]) for example and boundary value problems for ordinary differential equations (see, [1, 3, 9, 12, 16, 21] for example). More recently, quasilinearization has become a useful tool in the study of initial value problems for fractional differential equations; see [7, 26, 27]. Khan [15] has applied the quasilinearization method to nonlocal boundary value problem for fractional equations. Quasilinearization has also proved to be fruitful for boundary value problems at resonance for ordinary differential equations, [28, 25].

Recently, in [2], the shift argument [14] was applied to a boundary value problem at resonance for an ordinary differential equation in order that a quasilinearization method could be applied. In this article, we show that the construction produced in [2] can be modified to apply to a boundary value problem at resonance for a Caputo fractional differential equation. The method of upper and lower solutions is a primary tool and so a recent result due to Al-Refai [4] addressing the sign of fractional derivatives of a function at an extreme point plays a key role in the modification produced here. The motivation and development here is different than that in [25] or [28]; uniqueness of solutions is a key feature in this work and multiplicity of solutions is key in methods discussed in [28] or [25]. The problem studied here has also been recently studied in [11] for a boundary value problem at resonance for a Riemann-Liouville fractional differential equation.

In what follows, in Section 2 we provide preliminary definitions and we introduce and modify the work found in [4] to address extreme points of functions satisfying differential inequalities. In Section 3, we employ the method of upper and lower solutions and under suitable hypotheses obtain the uniqueness of solutions of a two-point fractional boundary value problem at resonance for a Caputo fractional differential equation for $1 < \alpha < 2$. Then we apply the shift argument and obtain existence of that unique solution. In Section 4, we construct a sequence of upper solutions and lower solutions that converge monotonically to the unique solution, and then employ the shift argument to obtain a quadratic rate of convergence. We close in Section 5 with two examples in which explicit upper and lower solutions are explicitly exhibited.

2 Preliminaries

Definition 2.1 *Let $0 < \alpha$. For $a \in \mathbb{R}$, the α-th Riemann-Liouville fractional integral of a function, y, is defined by*

$$I_a^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s)ds, \quad a \le t,$$

provided the right-hand side exists. For $\alpha = 0$, define I_a^α to be the identity map. Moreover, let n denote a positive integer and assume $n-1 < \alpha \leq n$. The Riemann-Liouville fractional derivative of order α is defined as

$$D_a^\alpha y(t) = D^n I_a^{n-\alpha} y(t), \quad a \leq t,$$

where D^n denotes the classical nth order derivative, if the right-hand side exists. If a function y is such that

$$D_a^\alpha \Big(y(t) - \sum_{i=0}^{n-1} y^{(i)}(a) \frac{(t-a)^i}{i!} \Big)$$

exists, then the Caputo fractional derivative of order α of the function y is defined by

$$D_{Ca}^\alpha y(t) = D_a^\alpha \Big(y(t) - \sum_{k=0}^{n-1} y^{(i)}(a) \frac{(t-a)^i}{i!} \Big).$$

If y^{n-1} is an absolutely continuous function, then

$$D_{Ca}^\alpha y(t) = I_a^{n-\alpha} D^n y(t), \quad a.e. \tag{1}$$

Again for $\alpha = 0$, define D_{Ca}^α to be the identity map.

The following theorem is stated and proved (for a minimum value) in [4]. It is an important result for the application of upper and lower solutions to fractional differential equations.

Theorem 2.1 *Assume $y \in C^2[0,1]$ attains its maximum value at $t_0 \in (0,1)$. Then for all $1 < \alpha < 2$,*

$$D_{C0}^\alpha y(t_0) \leq \frac{t_0^{-\alpha}}{\Gamma(2-\alpha)} \Big[(\alpha - 1)(y(0) - y(t_0)) - t_0 y^{'}(0) \Big]. \tag{2}$$

We have need of weaker hypotheses as stated in the next theorem. In [20] and [24], it is shown that solutions for boundary value problems can satisfy (2) under weaker hypotheses. We modify the proofs here to adapt to solutions of the boundary value problem we consider in Section 3.

Theorem 2.2 *Assume $1 < \alpha < 2$. Assume $y \in C^1[0,1]$, assume $D_{C0}^\alpha y \in C^1[0,1]$, and assume $y'(0) = 0$. Then y' is absolutely continuous on $[0,1]$. Moreover, for each $0 < \epsilon < 1$, $y \in C^2[\epsilon, 1]$.*

Proof: Begin with the definition

$$D_{C0}^\alpha y(t) = D_0^\alpha (y(t) - y(0) - y'(0)t) = D^2 I_0^{2-\alpha} (y(t) - y(0) - y'(0)t).$$

Since $y \in C^1[0,1]$ an integration by parts is valid and

$$D_{C0}^\alpha y(t) = D^2 I_0^{3-\alpha} (y'(t) - y'(0)) = D I_0^{2-\alpha} y'(t).$$

Thus,

$$\int_0^t D_{C0}^{\alpha} y(s)ds = I_0^{2-\alpha} y'(t),$$

which implies

$$\begin{aligned} y'(t) &= D_0^{2-\alpha} I_0^{2-\alpha} y'(t) = D_0^{2-\alpha} \int_0^t D_{C0}^{\alpha} y(s)ds = D I_0^{\alpha-1} \int_0^t D_{C0}^{\alpha} y(s)ds \\ &= D I_0^{\alpha-1} I D_{C0}^{\alpha} y(t) = D I I_0^{\alpha-1} D_{C0}^{\alpha} y(t) = I_0^{\alpha-1} D_{C0}^{\alpha} y(t). \end{aligned}$$

In particular,

$$y'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} D_{C0}^{\alpha} y(s)ds. \tag{3}$$

Since $D_{C0}^{\alpha} y \in C[0,1]$, the integrand in (3) is $L_1[0,1]$ and y' is an absolutely continuous function on $[0,1]$.

Since y' is an absolutely continuous function on $[0,1]$, a Taylor expansion for Caputo derivatives (Corollary 3.9 [8]) is valid for y and

$$y(t) = y(0) + y'(0)t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} D_{C0}^{\alpha} y(s)ds.$$

Since $D_{C0}^{\alpha} y \in C[0,1]$, an integration by parts is allowed and

$$y(t) = y(0) + y'(0)t + \frac{(D_{C0}^{\alpha} y)(0)}{\Gamma(\alpha+1)} t^{\alpha} + \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha} D D_{C0}^{\alpha} y(s)ds. \tag{4}$$

For $0 < t \leq 1$, differentiate (4) twice to obtain

$$y''(t) = \frac{(D_{C0}^{\alpha} y)(0)}{\Gamma(\alpha-1)} t^{\alpha-2} + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} D D_{C0}^{\alpha} y(s)ds. \tag{5}$$

Finally, since $D D_{C0}^{\alpha} y \in C[0,1]$ and $(t-s)^{\alpha-2}$ is absolutely integrable on $[0,t]$, the dominated convergence theorem implies that y'' is continuous for each $t \in (0,1]$. □

Theorem 2.3 *Assume $1 < \alpha < 2$. Assume $y \in C^1[0,1]$, assume $D_{C0}^{\alpha} y \in C^1[0,1]$, and assume $y'(0) = 0$. Assume y attains its maximum value at $t_0 \in (0,1]$. Assume further if $t_0 = 1$, then $y'(1) = 0$. Then*

$$D_{C0}^{\alpha} y(t_0) \leq \frac{t_0^{-\alpha}}{\Gamma(2-\alpha)} \Big[(\alpha-1)(y(0) - y(t_0)) \Big] \leq 0. \tag{6}$$

Proof: Apply Theorem 2.2 and note that y' is absolutely continuous on $[0,1]$ and for each $\epsilon > 0$, $y'' \in C[\epsilon, 1]$. Thus, the form (1) can be employed for $0 < t$ and the proof developed by Al-Refai to obtain Theorem 2.1 can now be applied directly to prove Theorem 2.3. □

Next, we obtain a version of Taylor's theorem with remainder for Caputo fractional derivatives. The proof we offer is motivated by work produced in [8].

Theorem 2.4 *Assume* $1 < \alpha < 2$. *Assume* $y \in C^1[0,1]$ *and* $D^{\alpha}_{C0}y \in C^1[0,1]$. *Then for each* $t \in (0,1)$, *there exists* $c \in (0,t)$ *such that*

$$y(t) = y(0) + y^{'}(0)t + \frac{D^{\alpha}_{C0}y(c)}{\Gamma(\alpha+1)}t^{\alpha}. \tag{7}$$

Proof: The relation between initial value problems and fixed point integral operators is known, and has been proved for $0 < \alpha < 1$ in [8] and for $n-1 < \alpha < n$, n a positive integer in [10]; in particular, if $y(t) \in C^1[0,1]$ and $D^{\alpha}_{C0}y(t) \in C[0,1]$, then

$$y(t) = y(0) + y^{'}(0)t + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}D^{\alpha}_{C0}y(s)ds. \tag{8}$$

The mean value theorem for integrals can be applied directly to (8) to obtain (7). □

3 Uniqueness of solutions and existence of solutions

Let $1 < \alpha < 2$ and assume throughout that $f = f(t,y) : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $\frac{\partial}{\partial t}f = f_t : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, and $\frac{\partial}{\partial y}f = f_y : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. We consider the two point boundary value problem for a Caputo fractional differential equation,

$$D^{\alpha}_{C0}y(t) = f(t, y(t)), \quad 0 \le t \le 1, \tag{9}$$

$$y'(0) = 0, \quad y'(1) = 0. \tag{10}$$

The boundary value problem (9), (10) is at resonance since constant functions are solutions of the homogeneous problem $D^{\alpha}_{C0}y(x) = 0$ and satisfy the boundary conditions (10).

Remark 3.1 *We shall apply Theorem 2.3 and Theorem 2.4 repeatedly in this section. In order to do so, first note that if* $y \in C[0,1]$ *is a solution of the boundary value problem* (9), (10), *then* $D^{\alpha}_{C0}y \in C[0,1]$ *and*

$$y(t) = y(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y(s))ds = y(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}D^{\alpha}_{C0}y(s)ds.$$

This implies that (Theorem 6.25, [8]) that $y \in C^1[0,1]$, *which implies, under the assumption that* f, f_t, f_y *are continuous on* $[0,1]\times\mathbb{R}$, *if* $y \in C[0,1]$ *is a solution of the boundary value problem* (9), (10), *then* $y \in C^1[0,1]$ *and* $D^{\alpha}_{C0}y \in C^1[0,1]$; *in particular, the results of Section 2 apply to continuous solutions of the boundary value problem* (9), (10).

We begin with the assumption that f is increasing in the second component and obtain results for the uniqueness of solutions. In the case of second order ordinary differential equations, this is a standard assumption to obtain uniqueness of solutions.

Theorem 3.1 *Assume* f, f_t, f_y *are continuous on* $[0,1]\times\mathbb{R}$, *and assume* $f_y > 0$ *on* $[0,1]\times\mathbb{R}$. *Then continuous solutions of the boundary value problem* (9), (10) *are unique, if they exist.*

Proof: Assume for the sake of contradiction that $y_1(t)$ and $y_2(t)$ denote two distinct continuous solutions of the boundary value problem (9), (10). So assume without loss of generality that $y_1 - y_2$ has a positive maximum at $t_0 \in [0,1]$.

First, assume, $t_0 \in (0,1]$. Then by Corollary 2.3 we have

$$D_{C0}^{\alpha}(y_1 - y_2)(t_0) \le 0.$$

However, y_1 and y_2 satisfy (9), and

$$D_{C0}^{\alpha}(y_1 - y_2)(t_0) = f(t_0, y_1(t_0)) - f(t_0, y_2(t_0)) > 0.$$

since f is increasing in y. Thus, $y_1 - y_2$ does not have a positive maximum at $t_0 \in (0,1]$.

Second, assume $t_0 = 0$ and recall from (10), $y_1^{'}(0) = y_2^{'}(0) = 0$. By Theorem 2.4, there exists $c \in (0,t)$ such that

$$\begin{aligned}(y_1 - y_2)(t) &= (y_1 - y_2)(0) + (y_1 - y_2)^{'}(0)t + D_{C0}^{\alpha}(y_1 - y_2)(c)\frac{t^{\alpha}}{\Gamma(\alpha+1)}\\ &= (y_1 - y_2)(0) + (f(c, y_1(c)) - f(c, y_2(c)))\frac{t^{\alpha}}{\Gamma(\alpha+1)}\\ &> (y_1 - y_2)(0),\end{aligned}$$

for $0 < t$ sufficiently small. Thus, $y_1 - y_2$ does not have a positive maximum at $t_0 = 0$.

Conclude that $y_1(t) \le y_2(t)$ for all $t \in [0,1]$. Similarly, $y_2(t) \le y_1(t)$ for all $t \in [0,1]$ and so, continuous solutions of (9), (10) are unique if they exist. □

Definition 3.1 *We say w is a lower solution of the boundary value problem* (9), (10) *if $w \in C^1[0,1]$ and $D_{C0}^{\alpha}w \in C^1[0,1]$, $w'(0) = 0$ and $w'(1) = 0$, and*

$$D_{C0}^{\alpha}w(t) \ge f(t, w(t)), \quad 0 \le t \le 1.$$

We say v is an upper solution of the boundary value problem (9), (10) *if $v \in C^1[0,1]$ and $D_{C0}^{\alpha}v \in C^1[0,1]$, $v'(0) = 0$ and $v'(1) = 0$, and*

$$D_{C0}^{\alpha}v(t) \le f(t, v(t)), \quad 0 \le t \le 1.$$

Theorem 3.2 *Assume f, f_t, f_y are continuous on $[0,1]\times\mathbb{R}$, and assume $f_y > 0$ on $[0,1]\times\mathbb{R}$. Assume w is a lower solution of the boundary value problem* (9), (10) *and assume v is an upper solution of the boundary value problem* (9), (10)*. Then*

$$w(t) \le v(t), \quad 0 \le t \le 1.$$

Proof: The proof of this theorem is very similar to the proof of the uniqueness theorem, Theorem 3.1. Assume w is a lower solution and v is an upper solution of the boundary value problem (9), (10), respectively. Assume for the sake of contradiction that $w(t) \le v(t)$ is false. Assume that $w - v$ has positive maximum at $t_0 \in [0,1]$.

First, assume, $t_0 \in (0,1]$. Then by Corollary 2.3 we have

$$D_{C0}^{\alpha}(w - v)(t_0) \le 0.$$

However, w and v are lower and upper solutions, respectively of (9), (10) and so,

$$D_{C0}^{\alpha}(w-v)(t_0) \geq f(t_0, w(t_0)) - f(t_0, v(t_0)) > 0.$$

since f is increasing in y. Thus, $w-v$ does not have a positive maximum at $t_0 \in (0,1]$.

Second, assume $t_0 = 0$ and recall $w^{'}(0) = v^{'}(0) = 0$. By Theorem 2.4, there exists $c \in (0,t)$ such that

$$\begin{aligned}(w-v)(t) &= (w-v)(0) + (w-v)^{'}(0)t + D_{C0}^{\alpha}(w-v)(c)\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ &= (w-v)(0) + (f(c,w(c)) - f(c,v(c)))\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ &> (w-v)(0),\end{aligned}$$

for $0 < t$ sufficiently small. Thus, $w-v$ does not have a positive maximum at $t_0 = 0$. □

We now turn to the question of existence of solutions of the boundary value problem (9), (10). The shift argument [14] will be applied and to do so, an appropriate Green's function, employing Mittag - Leffler functions, is constructed. We employ definitions and properties that are commonly used and refer the reader to [22] or [13].

Definition 3.2 *Let $\alpha, \beta > 0$. A two-parameter function of the Mittag-Leffler type is defined by the series expansion given by*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

Lemma 3.3 *The following relations hold:*

1. $E_{\alpha,\beta}(z) := zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}$, *for* $\Re(\beta) > 0$;
2. $\left(\frac{d}{dz}\right)^k \left(z^{\beta-1}E_{\alpha,\beta}(z^{\alpha})\right) = z^{\beta-k-1}E_{\alpha,\beta-k}(z^{\alpha}), \quad k \in N.$

Let $\mathcal{L}\{y(t); s\} = Y(s)$ denote the Laplace transform of y.

Lemma 3.4 *The following relations hold:*

1. $\mathcal{L}\Big\{D_{C0}^{\alpha}y(t); s\Big\} = s^{\alpha}Y(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}y^{(k)}(0)$ *where* $n-1 < \alpha \leq n$, n *a positive integer;*
2. $\mathcal{L}\{t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(\pm at^{\alpha}); s\} = \frac{k!s^{\alpha-\beta}}{(s^{\alpha}\mp a)^{k+1}}, \quad \Re(s) > |a|^{\frac{1}{\alpha}}$, *where* $E_{\alpha,\beta}^{(k)}(y) \equiv \frac{d^k}{dy^k}E_{\alpha,\beta}(y)$.

To obtain existence of solutions, we shall apply the shift argument [14]. Assume $K \neq 0$ and consider the shifted equation

$$D_{C0}^{\alpha}y(t) - K^2y(t) = \hat{f}(t, y(t)) = f(t, y(t)) - K^2y(t), \quad 0 \leq t \leq 1. \tag{11}$$

The boundary value problem (11), (10) is not at resonance since Theorem 3.1 implies that $y \equiv 0$ is the only solution of the homogenous fractional problem

$$D_{C0}^{\alpha} y(t) = K^2 y(t) \tag{12}$$

satisfying the boundary conditions, (10) for any $K \neq 0$.

Since the boundary value problem (11), (10) is not at resonance, we shall construct the corresponding Green's function of the shifted boundary value problem. To do so, apply the Laplace transform to

$$D_{C0}^{\alpha} y(t) - K^2 y(t) = h(t), \quad y'(0) = 0, y'(1) = 0.$$

Then

$$\mathcal{L}\{D_{C0}^{\alpha} y(t); s\} - K^2 \mathcal{L}\{y(t); s\} = \mathcal{L}\{h(t); s\}.$$

Let $H(s) = \mathcal{L}\{h(t); s\}$. By Lemma 3.4 and well-known properties of the Laplace transform,

$$s^{\alpha} Y(s) - \sum_{k=0}^{1} s^{\alpha-k-1} y(t)^{(k)}|_{t=0} - K^2 Y(s) = H(s).$$

Thus,

$$Y(s) = \frac{y(0)s^{\alpha-1}}{s^{\alpha} - K^2} + \frac{H(s)}{s^{\alpha} - K^2}.$$

Apply the inverse Laplace transform as indicated in Lemma 3.4 (2) to obtain

$$y(t) = y(0)E_{\alpha,1}(K^2 t^{\alpha}) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(K^2 (t-s)^{\alpha}) h(s) ds.$$

Apply Lemma 3.3 (1) with $\beta = 1$ and $z = K^2 t^{\alpha}$ to obtain

$$y(t) = y(0)\Big(K^2 t^{\alpha} E_{\alpha,\alpha+\beta}(K^2 t^{\alpha}) + \frac{1}{\Gamma(1)}\Big) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(K^2 (t-s)^{\alpha}) h(s) ds.$$

Since $y'(1) = 0$,

$$y(0) = \frac{-1}{K^2 E_{\alpha,\alpha}(K^2)} \int_0^1 (1-s)^{\alpha-2} E_{\alpha,\alpha-1}(K^2(1-s)^{\alpha}) h(s) ds.$$

Thus,

$$\begin{aligned} y(t) = \int_0^t \Big(& \frac{-(1-s)^{\alpha-2} E_{\alpha,\alpha-1}(K^2(1-s)^{\alpha}) E_{\alpha,1}(K^2 t^{\alpha})}{K^2 E_{\alpha,\alpha}(K^2)} \\ & + (t-s)^{\alpha-1} E_{\alpha,\alpha}(K^2(t-s)^{\alpha})\Big) h(s) ds \\ & + \int_t^1 \frac{-(1-s)^{\alpha-2} E_{\alpha,\alpha-1}(K^2(1-s)^{\alpha}) E_{\alpha,1}(K^2 t^{\alpha})}{K^2 E_{\alpha,\alpha}(K^2)} h(s) ds. \end{aligned}$$

Set

$$g(K,t,s)=\frac{-(1-s)^{\alpha-2}E_{\alpha,\alpha-1}(K^2(1-s)^{\alpha})E_{\alpha,1}(K^2t^{\alpha})}{K^2E_{\alpha,\alpha}(K^2)}.$$

Then the solution y has the form

$$y(t)=\int_0^1 G(K;t,s)h(s)ds$$

where

$$G(K;t,s)=\begin{cases} g(K,t,s)+(t-s)^{\alpha-1}E_{\alpha,\alpha}(K^2(t-s)^{\alpha}), & 0\le s<t\le 1,\\ \\ g(K,t,s), & 0\le t\le s<1. \end{cases} \tag{13}$$

We derive two standard properties of the Green's function, $G(K;t,s)$. First note that for K sufficiently small

$$G(K;t,s)<0, \quad (t,s)\in(0,1)\times(0,1). \tag{14}$$

To see this, let

$$H(K,t,s)=(t-s)^{\alpha-1}E_{\alpha,\alpha}(K^2(t-s)^{\alpha})$$

and write

$$G(K;t,s)=\begin{cases} g(K,t,s)+H(K,t,s), & 0\le s<t\le 1,\\ \\ g(K,t,s), & 0\le t\le s<1. \end{cases}$$

Note that for $0\le t\le s<1$, $g(K,t,s)\le 0$. To see that $g(K,t,s)+H(K,t,s)\le 0$ for $0<s<t<1$ and for K sufficiently small, note that

$$\lim_{K\to 0}K^2E_{\alpha,\alpha}(K^2)\Big((g(K,t,s)+H(K,t,s)\Big)=-(1-s)^{\alpha-2}\le -1, \quad 0\le s<t\le 1.$$

Second, we bound $\max_{0\le t\le 1}\int_0^1|G(K;t,s)|ds$. Note that

$$g(K,t,s)\le g(K,t,s)+H(K,t,s)\le 0, \quad 0\le s<t\le 1$$

implies

$$g(K,t,s)\le G(K;t,s)\le 0, \quad (t,s)\in(0,1)\times(0,1).$$

Thus,

$$\max_{0\le t\le 1}\int_0^1|G(K;t,s)|ds\le \max_{0\le t\le 1}\int_0^1|g(K,t,s)|ds.$$

Let

$$A=\frac{E_{\alpha,1}(K^2t^{\alpha})}{K^2E_{\alpha,\alpha}(K^2)}.$$

Then,

$$\begin{aligned}\int_0^t |g(k,t,s)|ds &= A\int_0^1 (1-s)^{\alpha-2}E_{\alpha,\alpha-1}(K^2(1-s)^{\alpha})ds\\ &= A\int_0^t \sum_{k=0}^{\infty}\frac{K^{2k}(1-s)^{k\alpha+\alpha-2}}{\Gamma(k\alpha+\alpha-1)}ds\\ &= A\left[\sum_{k=0}^{\infty}\frac{-K^{2k}(1-s)^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha-1)(k\alpha+\alpha)}\right]_0^1 = AE_{\alpha,\alpha}(K^2) = \frac{E_{\alpha,1}(K^2t^{\alpha})}{K^2}.\end{aligned}$$

Thus,

$$\max_{0\le t\le 1}\int_0^1 |G(K;t,s)|ds \le \max_{0\le t\le 1}\int_0^1 |g(K,t,s)|ds = \max_{0\le t\le 1}\frac{E_{\alpha,1}(K^2t^{\alpha})}{K^2} = \frac{E_{\alpha,1}(K^2)}{K^2}. \quad (15)$$

Theorem 3.5 *Assume f, f_t, f_y are continuous on $[0,1]\times\mathbb{R}$, and assume there exists $K^2>0$ such that $f_y \ge K^2 > 0$ on $[0,1]\times\mathbb{R}$. Assume w and v are lower and upper solutions of the boundary value problem (9), (10), respectively. Then there exists a unique continuous solution y of (9), (10) satisfying*

$$w(t)\le y(t)\le v(t), \quad 0\le t\le 1.$$

Proof: Let $K\ne 0$ and define a continuously differentiable truncation of $\hat{f}(t,y(t)) = f(t,y)-K^2y$ as follows. First, set

$$\begin{aligned}l(t,y,z) &= \frac{(f(t,y)-K^2y)-(f(t,z)-K^2z)}{1+(f(t,y)-K^2y)-(f(t,z)-K^2z)}\\ &= 1-\frac{1}{1+(f(t,y)-K^2y)-(f(t,z)-K^2z)}.\end{aligned}$$

Define the truncation, F, by

$$F(t,y(t)) = \begin{cases} f(t,v(t))-K^2v(t)+l(t,y(t),v(t)), & \text{if } y(t)>v(t),\\ f(t,y(t))-K^2y(t), & \text{if } w(t)\le y(t)\le v(t),\\ f(t,w(t))-K^2w(t)+l(t,y(t),w(t)), & \text{if } y(t)<w(t).\end{cases}$$

Note that if $y(t)>v(t)$, there exists $v(t)<c(t)<y(t)$ such that

$$(f(t,y(t))-K^2y(t))-(f(t,v(t))-K^2v(t)) = (f_y(t,c(t))-K^2)(y(t)-v(t))>0.$$

Thus, for $y(t)>v(t)$,

$$1+(f(t,y(t))-K^2y(t))-(f(t,v(t))-K^2v(t))\ge 1$$

and $l(t,y(t),v(t))$ is bounded. Similarly, for $y(t)<w(t)$, $l(t,y(t),w(t))$ is bounded, and so F is bounded.

To address the continuous differentiability of F, note that

$$l(t, v(t), v(t)) = l(t, w(t), w(t)) = 0.$$

Thus the truncation F is continuous. Note that for $y(t) > v(t)$,

$$\frac{d}{dt}l(t, y(t), v(t)) = \frac{(f_t(t, y(t)) + f_y(t, y(t)) - K^2)y'(t) - (f_t(t, v(t)) + f_y(t, v(t)) - K^2)v'(t)}{(1 + (f(t, y(t)) - K^2y(t)) - (f(t, v(t)) - K^2v(t)))^2}$$

and

$$\begin{aligned}\lim_{y(t)\to v(t)^+} \frac{d}{dt}l(t, y(t), v(t)) = (f_t(t, y(t)) + f_y(t, y(t)) - K^2)y'(t) \\ - (f_t(t, v(t)) + f_y(t, v(t)) - K^2)v'(t).\end{aligned}$$

In particular,

$$\begin{aligned}\lim_{y(t)\to v(t)^+} \frac{d}{dt}F(t, y(t)) &= \lim_{y(t)\to v(t)^+}((f_t(t, v(t)) + f_y(t, v(t)) - K^2)v'(t) \\ &+ (f_t(t, y(t)) + f_y(t, y(t)) - K^2)y'(t) - (f_t(t, v(t)) + f_y(t, v(t)) - K^2)v'(t) \\ &= (f_t(t, y(t)) + f_y(t, y(t)) - K^2)y'(t) = \lim_{y(t)\to v(t)^-} \frac{d}{dt}F(t, y(t)).\end{aligned}$$

Similarly, it is shown that

$$\lim_{y(t)\to w(t)^-} \frac{d}{dt}F(t, y(t)) = \lim_{y(t)\to w(t)^+} \frac{d}{dt}F(t, y(t))$$

and the truncation F is continuously differentiable.

Define an operator $T : C[0, 1] \to C[0, 1]$ by

$$Ty(t) = \int_0^1 G(K; t, s)F(s, y(s))ds$$

where $G(K; t, s)$ is given by (13). The following statement is an important consequence of Remark 3.1. Then $y \in C^1[0, 1]$, $D^{\alpha}_{C0}y \in C^1[0, 1]$, is a solution of the boundary value problem

$$D^{\alpha}_{C0}y(t) - K^2y(t) = F(t, y(t)), \quad 0 \le t \le 1, \quad y'(0) = y'(1) = 0 \tag{16}$$

if, and only if, $y \in C[0, 1]$, and

$$y(t) = \int_0^1 G(K; t, s)F(s, y(s))ds, \quad 0 \le t \le 1.$$

In particular, in fixed point theorem applications, we can choose the Banach space, $C[0, 1]$.

Note that the truncation F is bounded and continuous on $[0, 1] \times \mathbb{R}$. So it is a straightforward application of the Schauder fixed point theorem to show that the boundary value problem (16) has a solution, $y \in C[0, 1]$. To see this, let

$$M = \sup\{|F(t, y)| : 0 \le t \le 1, y \in \mathbb{R}\},$$

and let
$$\max_{0\le t\le 1}\int_0^1 |G(K;t,s)|ds \le \frac{E_{\alpha,1}(K^2)}{K^2} = G.$$
Then if $y \in C[0,1]$,
$$||Ty|| = \max_{0\le t\le 1} |\int_0^1 G(K;t,s)F(s,y(s))ds| \le MG,$$
where $||\cdot||$ denotes the supremum norm on $[0,1]$.

Define
$$\mathcal{U} = \{y \in C[0,1] : ||y|| \le MG\}.$$
Then $\mathcal{U}$ is a closed convex subset of $C[0,1]$ and $T : \mathcal{U} \to \mathcal{U}$. It can be shown that T on $C[0,1]$ is a completely continuous map and so, the Schauder fixed point theorem implies there exists a fixed point, $y \in \mathcal{U}$ of the operator T.

Let y denote a fixed point of the operator T. Then
$$D^{\alpha}_{C0}y(t) = F(t,y(t)) + K^2y(t), \quad y'(0) = y'(1) = 0.$$
It follows by Remark 3.1 that $y \in C^1[0,1]$ and $D^{\alpha}_{C0}y \in C^1[0,1]$. Thus, Theorem 2.3 and Theorem 2.4 can be employed to show
$$w(t) \le y(t) \le v(t), \quad 0 \le t \le 1.$$
Then by the definition of the truncation F it follows that the fixed point y of T is a solution of the original boundary value problem (9), (10).

We show the details that $y - v$ does not have a positive maximum at $t_0 \in [0,1]$. First, assume for the sake of contradiction that $y - v$ has a positive maximum at $t_0 \in (0,1]$. Then
$$D^{\alpha}_{C0}y(t_0) - K^2y(t_0) = f(t_0,v(t_0)) - K^2v(t_0) + l(t_0,y(t_0),v(t_0)).$$
Note that the condition, $f_y \ge K^2 > 0$ on $[0,1]\times\mathbb{R}$ implies $l(t_0,y(t_0),v(t_0)) \ge 0$ if $y(t_0 > v(t_0)$. Since v is an upper solution of (9), (10), it follows that
$$\begin{aligned} D^{\alpha}_{C0}(y-v)(t_0) &\ge (f(t_0,v(t_0)) - f(t_0,v(t_0))) + K^2(y(t_0)-v(t_0)) \\ &\quad + l(t_0,y(t_0),v(t_0)) > 0. \end{aligned}$$
This contradicts Theorem 2.3 and so, $y-v$ does not have a positive maximum at $t_0 \in (0,1]$.

The argument to show $y - v$ does not have a positive maximum at $t_0 = 0$ follows analogously to the proof in Theorem 3.1 and so,
$$y(t) \le v(t), \quad 0 \le t \le 1.$$

The arguments to show
$$w(t) \le y(t), \quad 0 \le t \le 1$$
are completely analogous.

To close, since
$$w(t) \le y(t) \le v(t), \quad 0 \le t \le 1,$$
then $F(t,y(t)) = f(t,y(t)) - K^2y(t)$ and y is a solution of the boundary value problem (9), (10). □

4 The monotone method and quadratic convergence

In this section, we describe the monotone method and obtain a quadratic rate of convergence. Now that the uniqueness and existence results are obtained in Section 3, the details presented in this section are primarily standard; see, [12] or [17]. Thus, we outline the construction.

Theorem 4.1 *Assume f, f_t, f_y are continuous on $[0,1]\times\mathbb{R}$, and assume there exists $K^2 > 0$ such that $f_y \geq K^2 > 0$ on $[0,1]\times\mathbb{R}$. In addition, assume $f_{yy} \geq 0$ on $[0,1]\times\mathbb{R}$. Assume w_0 and v_0 are lower and upper solutions of the boundary value problem* (9), (10), *respectively. Then there exists a unique continuous solution y of* (9), (10) *satisfying*

$$w_0(t) \leq y(t) \leq v_0(t), \quad 0 \leq t \leq 1.$$

Moreover, there exist sequences $\{w_n\}, \{v_n\}$ of lower and upper solutions of the boundary value problem (9), (10), *respectively, each of which converges quadratically in $C[0,1]$ to the unique solution y of the boundary value problem* (9), (10) *and satisfy*

$$w_n(t) \leq w_{n+1}(t) \leq y(t) \leq v_{n+1}(t) \leq v_n(t), \quad 0 \leq t \leq 1, \quad n = 0, 1, \ldots.$$

*Proof:*Let w_0, v_0 denote a lower and an upper solution of (9), (10), respectively. So, under the assumption that $f_y(t,y) > 0$ on $[0,1]\times\mathbb{R}$,

$$w_0(t) \leq v_0(t), \quad 0 \leq t \leq 1.$$

Define the function $h(t; w_0, v_0)$ on $[0,1]$ by

$$h(w_0, v_0; t, y) = f(t, w_0(t)) + f_y(t, v_0(t))(y - w_0)(t)$$

and consider the boundary value problem for the linear nonhomogeneous fractional differential equation

$$D_{C0}^{\alpha} y(t) = h(w_0, v_0; t, y(t)), \quad 0 \leq t \leq 1, \quad y'(0) = 0, \quad y'(1) = 0. \tag{17}$$

Note that h, h_t, h_y are continuous on $[0,1]\times\mathbb{R}$ and so, Theorem 2.3 can be applied to continuous solutions, smooth lower solutions, and smooth upper solutions of the boundary value problem (17). Also note that

$$h(w_0, v_0; t, w_0(t)) = f(t, w_0(t)), \quad 0 \leq t \leq 1,$$

and so,

$$D_{C0}^{\alpha} w_0(t) \geq f(t, w_0(t)) = h(w_0, v_0; t, w_0(t)) \quad 0 \leq t \leq 1. \tag{18}$$

Moreover, there exists $c(t)$ satisfying $w_0(t) \leq c(t) \leq v_0(t)$ such that

$$f(t, v_0(t)) = f(t, w_0(t)) + f_y(t, c(t))(v_0 - w_0)(t).$$

Hence,

$$\begin{aligned} f(t, w_0(t)) + f_y(t, c(t))(v_0 - w_0)(t) &\leq f(t, w_0(t)) + f_y(t, v_0(t))(v_0 - w_0)(t) \\ &= h(w_0, v_0; t, v_0(t)) \quad 0 \leq t \leq 1, \end{aligned}$$

since f_y is increasing in y for each $t \in [0,1]$. Thus,

$$h(w_0, v_0; t, v_0(t)) \geq f(t, v_0(t)) \geq D^{\alpha}_{C0} v_0(t), \quad 0 \leq t \leq 1. \tag{19}$$

In particular, (18) and (19) imply w_0, v_0 are lower and upper solutions of (17) respectively as well. Since, h also satisfies the hypotheses of Theorem 3.5, there exists a solution, $w_1(t)$, of (17) satisfying

$$w_0(t) \leq w_1(t) \leq v_0(t), \quad 0 \leq t \leq 1.$$

Note that there exists $w_0(t) \leq c(t) \leq w_1(t) \leq v_0(t)$ such that

$$f(t, w_1(t)) - f(t, w_0(t)) = f_y(t, c(t))(w_1(t) - w_0(t)) \leq f_y(t, v_0(t))(w_1(t) - w_0(t))$$

and so,

$$D^{\alpha}_{C0} w_1(t) = h(w_0, v_0; t, w_1(t)) \geq f(t, w_1(t)), \quad 0 \leq t \leq 1.$$

In particular, w_1 is a lower solution (9), (10) since $w_1 \in C^1[0,1]$ and $D^{\alpha}_{C0} w_1 \in C^1[0,1]$.

Now define the function $k(v_0; t, y)$ on $[0,1]$ by

$$k(v_0; t, y) = f(t, v_0(t)) + f_y(t, v_0(t))(y - v_0)(t)$$

and consider the boundary value problem for the linear nonhomogeneous fractional differential equation

$$D^{\alpha}_{C0} y(t) = k(v_0; t, y(t)), \quad 0 \leq t \leq 1, \quad y'(0) = 0, \quad y'(1) = 0. \tag{20}$$

Note that

$$k(v_0; t, v_0(t)) = f(t, v_0(t)), \quad 0 \leq t \leq 1,$$

and

$$D^{\alpha}_{C0} v_0(t) \leq f(t, v_0(t)) = k(v_0; t, v_0(t)), \quad 0 \leq t \leq 1.$$

Thus, v_0 is an upper solution of (20). Note that there exists $c(t)$ satisfying $w_0(t) \leq c(t) \leq v_0(t)$ such that

$$\begin{aligned} D^{\alpha}_{C0} w_0(t) &\geq f(t, w_0(t)) = f(t, v_0(t)) + f_y(t, c(t))(w_0(t) - v_0(t)) \\ &\geq f(t, v_0(t)) + f_y(t, v_0(t))(w_0(t) - v_0(t)) = k(v_0; t, w_0(t)), \quad 0 \leq t \leq 1, \end{aligned}$$

and so, w_0 is a lower solution of (20). Since k satisfies the hypotheses of Theorem 3.5 there exists a solution, $v_1(t)$, of (20) satisfying

$$w_0(t) \leq v_1(t) \leq v_0(t), \quad 0 \leq t \leq 1.$$

An application of the mean value theorem again will give,

$$k(v_0; t, v_1(t)) \leq f(t, v_1(t)), \quad 0 \leq t \leq 1.$$

To see this, for some $v_1(t) \leq c(t) \leq v_0(t)$,

$$\begin{aligned} f(t, v_1(t)) &= f(t, v_0(t)) + f_y(t, c(t))(v_1(t) - v_0(t)) \\ &\geq f(t, v_0(t)) + f_y(t, v_0(t))(v_1(t) - v_0(t)). \end{aligned}$$

Thus,

$$D^{\alpha}_{C0}v_1(t) = k(v_0; t, v_1(t)) \le f(t, v_1(t)), \quad 0 \le t \le 1,$$

and v_1 is an upper solution of (9), (10) since $v_1 \in C^1[0,1]$ and $D^{\alpha}_{C0}v_1 \in C^1[0,1]$.

Finally, apply Theorem 3.2 to obtain

$$w_1(t) \le v_1(t), \quad 0 \le t \le 1;$$

in particular,

$$w_0(t) \le w_1(t) \le v_1(t) \le v_0(t), \quad 0 \le t \le 1.$$

Apply Theorem 3.5 with lower and upper solutions, w_1 and v_1, respectively, and keeping in mind that the solution y obtained in Theorem 3.5 is unique, to obtain

$$w_0(t) \le w_1(t) \le y(t) \le v_1(t) \le v_0(t), \quad 0 \le t \le 1,$$

where $y(t)$ is the unique solution of the boundary value problem (9), (10).

Assume the sequences $\{w_k\}_{k=1}^{n}$ and $\{v_k\}_{k=1}^{n}$ have been constructed inductively such that for each k

$$h(w_k, v_k; t, y) = f(t, w_k(t)) + f_y(t, v_k(t))(y - w_k)(t),$$
$$k(v_k; t, y) = f(t, v_k(t)) + f_y(t, v_k(t))(y - v_k)(t),$$

w_k is the solution of the boundary value problem

$$D^{\alpha}_{C0}y(t) = h(w_{k-1}, v_{k-1}; t, y(t)), \quad 0 \le t \le 1, \quad y'(0) = 0, \quad y'(1) = 0,$$

v_k is the solution of the boundary value problem

$$D^{\alpha}_{C0}y(t) = k(v_{k-1}; t, y(t)), \quad 0 \le t \le 1, \quad y'(0) = 0, \quad y'(1) = 0,$$

and

$$w_{k-1}(t) \le w_k(t) \le y(t) \le v_k(t) \le v_{k-1}(t), \quad 0 \le t \le 1,$$

$k = 0, \dots, n$, where w_k, v_k, $k = 1, \dots, n$ denote a lower solution and an upper solution, respectively of (9), (10), and y is the unique solution of the boundary value problem (9), (10).

To complete the induction argument, consider the boundary value problem for the linear nonhomogeneous fractional differential equation

$$D^{\alpha}_{C0}y(t) = h(w_n, v_n; t, y(t)), \quad 0 \le t \le 1, \quad y'(0) = 0, \quad y'(1) = 0. \tag{21}$$

Note that

$$h(w_n, v_n; t, w_n(t)) = f(t, w_n(t)), \quad 0 \le t \le 1,$$

and

$$h(w_n, v_n; t, v_n(t)) \ge f(t, v_n(t)), \quad 0 \le t \le 1.$$

So, w_n, v_n denote a lower and an upper solution of (21) respectively as well.

The arguments above to show the existence of $w_1(t)$ and $v_1(t)$ and the inequalities

$$w_0(t) \le w_1(t) \le v_1(t) \le v_0(t), \quad 0 \le t \le 1,$$

are readily adapted to show the existence of $w_{n+1}(t)$ and $v_{n+1}(t)$ and the inequalities

$$w_n(t) \leq w_{n+1}(t) \leq v_{n+1}(t) \leq v_n(t), \quad 0 \leq t \leq 1.$$

To complete the proof, $\{w_n\}$ and $\{v_n\}$ are monotone sequences of continuous functions bounded above or below, respectively, on a compact domain. So by Dini's theorem, each converges uniformly to $w(t), v(t)$ respectively on $[0,1]$.

$$k(v_n; t, v_{n+1}) = f(t, v_n(t)) + f_y(t, v_n(t))(v_{n+1} - v_n)(t) \to f(t, v)$$

uniformly on $[0,1]$ as $n \to \infty$. So $v = y$, the unique solution of (9), (10). Similarly,

$$h(w_n, v_n; t, w_{n+1}) = f(t, w_n(t)) + f_y(t, v_n(t))(w_{n+1} - w_n)(t) \to f(t, w)$$

as $n \to \infty$ and so, w is also the unique solution, y of (9), (10).

We now obtain an estimate on the error bound. For each n, define the error e_n as follows:

$$e_n(t) = v_n(t) - w_n(t), 0 \leq t \leq 1.$$

So, $0 \leq e_n(t)$ for $0 \leq t \leq 1$. Denote by $||e_n||$ the error bound

$$||e_n|| = \max_{0 \leq t \leq 1} |e_n(t)|.$$

Since $f_y(t, y) \geq K^2$, assume without loss of generality that K is sufficiently small and (14) holds; that is,

$$G(K; t, s) < 0, \quad (t, s) \in (0,1) \times (0,1),$$

where $G(K; t, s)$ is defined by (13).

Recall

$$\begin{aligned} D_{C0}^{\alpha} w_{n+1}(t) &= h(w_n, v_n; t, w_{n+1}(t)) = f(t, w_n(t)) + f_y(t, v_n(t))(w_{n+1}(t) - w_n(t)), \\ D_{C0}^{\alpha} v_{n+1}(t) &= k(v_n; t, v_{n+1}(t)) = f(t, v_n(t)) + f_y(t, v_n(t))(v_{n+1}(t) - v_n(t)). \end{aligned}$$

Then

$$\begin{aligned} D_{C0}^{\alpha} e_{n+1}(t) &= D_{C0}^{\alpha} v_{n+1}(t) - D_{C0}^{\alpha} w_{n+1}(t) \\ &= [f(t, v_n(t)) - f(t, w_n(t))] + f_y(t, v_n(t))[v_{n+1}(t) - v_n(t) - w_{n+1}(t) + w_n(t)] \\ &= [f(t, v_n(t)) - f(t, w_n(t))] + f_y(t, v_n(t))[e_{n+1}(t) - e_n(t)]. \end{aligned}$$

By the mean value theorem, there exists $c(t)$ satisfying $w_n(t) < c_n(t) < v_n(t)$ such that

$$f(t, v_n(t)) - f(t, w_n(t)) = f_y(t, c_n(t)) e_n(t).$$

Thus,

$$\begin{aligned} D_{C0}^{\alpha} e_{n+1}(t) &= f_y(t, c_n(t)) e_n(t) + f_y(t, v_n(t)) e_{n+1}(t) - f_y(t, v_n(t)) e_n(t) \\ &= f_y(t, v_n(t)) e_{n+1}(t) + [f_y(t, c_n(t)) - f_y(t, v_n(t))] e_n(t). \end{aligned}$$

Employ the mean value theorem again for $f_y(t, c_n(t)) - f_y(t, v_n(t))$ and there exists $\hat{c_n}(t)$ satisfying

$$c_n(t) < \hat{c_n}(t) < v_n(t)$$

such that

$$f_y(t, c_n(t)) - f_y(t, v_n(t)) = f_{yy}(t, \hat{c_n}(t))(c_n(t) - v_n(t)).$$

Then

$$D_{C0}^{\alpha} e_{n+1}(t) = f_y(t, v_n(t)) e_{n+1}(t) + f_{yy}(t, \hat{c_n}(t))(c_n(t) - v_n(t)) e_n(t).$$

Apply the shift argument and

$$D_{C0}^{\alpha} e_{n+1}(t) - K^2 e_{n+1}(t) = (f_y(t, v_n(t) - K^2) e_{n+1}(t) + f_{yy}(t, \hat{c_n}(t))(c_n(t) - v_n(t)) e_n(t).$$

Note that e_{n+1} satisfies the boundary conditions (10) and employ the Green's function (13). Then

$$\begin{aligned}
0 &\leq e_{n+1}(t) \\
&= \int_0^1 G(K; t, s) \left[(f_y(s, v_n(s)) - K^2) e_{n+1}(s) + f_{yy}(s, \hat{c_n}(s))(c_n(s) - v_n(s)) e_n(s) \right] ds \\
&\leq \int_0^1 |G(K; t, s)| f_{yy}(s, \hat{c_n}(s))(v_n(s) - c_n(s)) e_n(s) ds \\
&\leq \int_0^1 |G(K; t, s)| f_{yy}(s, \hat{c_n}(s)) e_n^2(s) ds, \qquad (22)
\end{aligned}$$

since

$$G(K; t, s)(f_y(s, v_n(s)) - K^2) e_{n+1}(s)) \leq 0, \quad 0 \leq s \leq 1.$$

Let

$$M = \max\{|f_{yy}(t, y)|, w_0(t) \leq y \leq v_0(t), 0 \leq t \leq 1\}$$

and recall (15)

$$\max_{0 \leq t \leq 1} \int_0^1 |G(K; t, s)| ds \leq \frac{E_{\alpha,1}(K^2)}{K^2} = G.$$

Then, from (22)

$$\begin{aligned}
|e_{n+1}(t)| &\leq \int_0^1 |G(K; t, s)| f_{yy}(s, \hat{c_n}(s)) e_n^2(s) ds \\
&\leq M \int_0^1 |G(K; t, s)| ds ||e_n||^2 \leq MG||e_n||^2,
\end{aligned}$$

and the rate of convergence is quadratic. □

5 Two examples

We close with two applications in which the shift method implies that constant lower and upper solutions can be exhibited.

Theorem 5.1 *Assume f, f_t, f_y are continuous on $[0,1]\times\mathbb{R}$, and assume there exists $K^2 > 0$ such that $f_y \geq K^2 > 0$ on $[0,1]\times\mathbb{R}$. In addition, assume $f_{yy} \geq 0$ on $[0,1]\times\mathbb{R}$. Assume K is sufficiently small so that*

$$G(K; t, s) < 0, \quad (t, s) \in (0,1)\times(0,1).$$

Assume there exist $\sigma \in C[0,1]$ and a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|f(t, y) - K^2 y| \leq \sigma(t)\psi(|y|), \quad (t, y) \in [0,1]\times\mathbb{R}$$

and there exists $M > 0$ such that

$$\frac{K^2 M}{||\sigma||_0 \psi(M)} > 1. \tag{23}$$

Then there exist sequences $\{w_n\}, \{v_n\}$ of lower and upper solutions, respectively, of the boundary value problem (9), (10), each of which converges quadratically in $C[0,1]$ to the unique solution y of the boundary value problem (9), (10) and satisfy

$$w_n(t) \leq w_{n+1}(t) \leq y(t) \leq v_{n+1}(t) \leq v_n(t), \quad 0 \leq t \leq 1.$$

Proof: Rather than assume the existence of v_0, one can employ the growth condition (23) to exhibit v_0, an upper solution. Set

$$v_0 = M.$$

Then

$$D^{\alpha}_{C0} v_0(t) - K^2 v_0 = -K^2 M \leq -||\sigma||_0 \psi(M) \leq f(t, v_0) - K^2 v_0.$$

To exhibit w_0 a lower solution, set

$$w_0 = -M.$$

Now apply Theorem 4.1. □

In a similar way, if the growth condition $|f(t, y) - K^2 y| \leq \sigma(t)\psi(|y|)$ is replaced by a boundedness condition, there exists $M > 0$ such that

$$|f(t, y) - K^2 y| \leq M, \quad (t, y) \in [0,1]\times\mathbb{R},$$

then upper and lower solutions are readily exhibited. Set $v_0 = \frac{M}{K^2}$ and set $w_0 = -v_0$.

Theorem 5.2 *Assume f, f_t, f_y are continuous on $[0,1]\times\mathbb{R}$, and assume there exists $K^2 > 0$ such that $f_y \geq K^2 > 0$ on $[0,1]\times\mathbb{R}$. In addition, assume $f_{yy} \geq 0$ on $[0,1]\times\mathbb{R}$. Assume K is sufficiently small so that*

$$G(K; t, s) < 0, \quad (t, s) \in (0,1)\times(0,1).$$

In addition, assume there exists $M > 0$ *such that*

$$|f(t,y) - K^2 y| \leq M, \quad (t,y) \in [0,1] \times \mathbb{R}.$$

Then there exist sequences $\{w_n\}, \{v_n\}$ *of lower and upper solutions, respectively, of the boundary value problem* (9)*,* (10)*, each of which converges quadratically in* $C[0,1]$ *to the unique solution* y*, of the boundary value problem* (9)*,* (10) *and satisfy*

$$w_n(t) \leq w_{n+1}(t) \leq y(t) \leq v_{n+1}(t) \leq v_n(t), \quad 0 \leq t \leq 1.$$

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